What is the Source of Fuzziness?

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Fuzziness is characteristic of the way people use natural languages. Over the centuries, philosophers, linguists, and logicians independently discovered and commented on many aspects of fuzziness, but without a common foundation for organizing and relating their discoveries. In their historical survey, Dubois, Ostasiewicz, and Prade (1999) cited numerous examples:

> Looking back in time, what is really amazing is the diversity of fields, where intuitions about fuzziness were expressed and more or less formalized, and the number of scientists who participated to the emergence of the fuzzy set concept. Also it is surprising to see how long it took before such a simple, although powerful, idea of graded membership, could be cast into a proper, widely accepted mathematical model, due to the far-ranged vision, the tenacity, and the numerous seminal papers of Lotfi Zadeh.

Dubois et al. presented a thorough survey of the mathematical methods for quantifying and computing with and about fuzziness. Zadeh (1996) identified fuzzy logic and "computing with words" (CWW). Mendel, Zadeh, and others (2010) discussed the challenge of relating the CWW methodology to the semantic issues in linguistics and the technology for natural language processing (NLP). This article surveys the issues and suggests some ways for relating them.

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1. Fuzziness in Language

According to Heraclitus, *panta rhei* — all things are in flux. But what gives that flux its form is the *logos* — the words or signs that enable us to perceive patterns in the flux, remember them, talk about them, and take action upon them even while we ourselves are part of the flux we are acting in and on. Modern physics is essentially a theory of flux in which the ultimate building blocks of matter maintain some semblance of stability only because of conservation laws of energy, momentum, spin, charge, and more exotic notions like charm and strangeness. Meanwhile, the concepts of everyday life are derived from experience with objects and processes that are measured and classified by comparisons with the human body, its parts, and its typical movements. Yet despite the vast differences in sizes, speeds, and time scale, the languages and counting systems of our stone-age ancestors have been successfully adapted to describe, analyze, and predict the behavior of everything from subatomic particles to clusters of galaxies that span the universe.

With such a vast range of topics, no language with a finite vocabulary can have a one-to-one mapping of words to every aspect of every topic. Vagueness is not only inevitable, it is necessary for language to be robust, flexible, and extensible. Dubois et al. cited the logician, philosopher, and scientist Charles Sanders Peirce as "one of the first scholars in the modern age" to point out the importance of vagueness. Peirce wrote a succinct summary of the issues:

> "It is easy to speak with precision upon a general theme. Only, one must commonly surrender all ambition to be certain. It is equally easy to be certain. One has only to be sufficiently vague. It is not so difficult to be pretty precise and fairly certain at once about a very narrow subject." (CP 4.237)

The narrow subjects for which precision is possible are ones that the speakers or authors selected for a specific purpose. In writing dictionary definitions, lexicographers start by defining the most typical examples, such as a chair with a back and four legs. Then they list exceptions that deviate from the type for various reasons. To illustrate that practice, Lehmann and Cohn (1994) drew egg-yolk diagrams such as Figure 1. Typical chairs are shown in the yolk, unusual chairs are in the egg white, and things that might be used as chairs are just outside the egg.

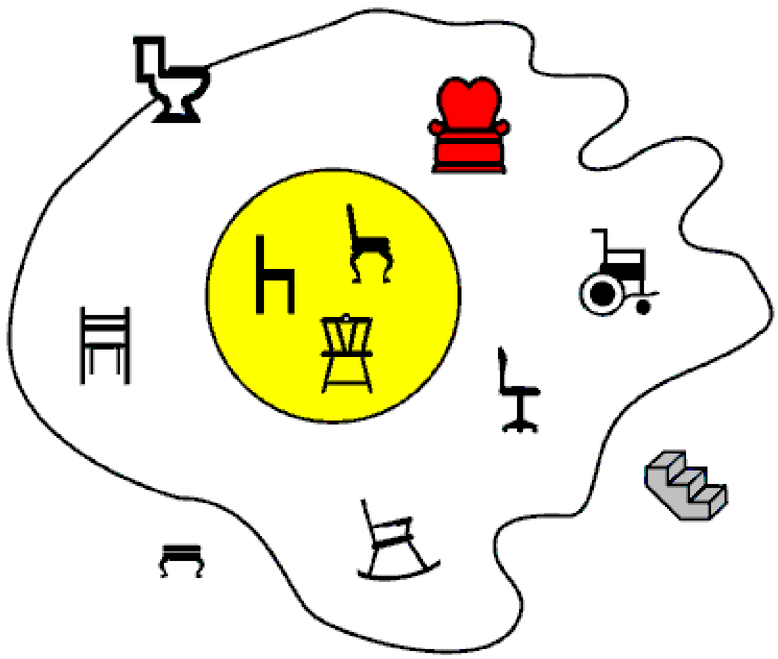

Figure 1. An egg-yolk diagram for the word *chair*

The boundaries of the egg and egg-yolk of Figure 1 resemble the way Bandler and Kohout (1988) partition a fuzzy set with *level cuts*. The level 0.9, for example, could determine the boundary of the yolk, which partitions the most typical chairs from the ones that omit or modify some typical characteristics. The level 0.6 could be the outer edge of the egg. The toilet on the edge of the egg would have that value. The footstool and the stairs, which are just outside the egg, would have values slightly less than 0.6. Yet those numbers by themselves cannot distinguish the significant differences between a folding chair, a rocking chair, a wheelchair, and a chair that has wheels at the bottom of a pod. The numbers are important for computing with words, but the reasons why those chairs differ from typical chairs are also important. Mendel (2010) noted "Numbers alone may not activate the CWW engine."

In the 19th century, William Whewell and John Stuart Mill debated the methods for representing and reasoning about variability. Whewell (1858) described the practice of biologists, who base their classifications on a type specimen for each species and a type species for each genus:

> Natural groups are given by Type, not by Definition. And this consideration accounts for that indefiniteness and indecision which we frequently find in the descriptions of such groups, and which must appear so strange and inconsistent to anyone who does not suppose these descriptions to assume any deeper ground of connection than an arbitrary choice of the botanist. Thus in the family of the rose tree, we are told that the ovules are very rarely erect, the stigmata usually simple. Of what use, it might be asked, can such loose accounts be? To which the answer is, that they are not inserted to distinguish the species, but in order to describe the family, and the total relations of the ovules and the stigmata of the family are better known by this general statement....

> Though in a Natural group of objects a definition can no longer be of any use as a regulative principle, classes are not therefore left quite loose, without any certain standard or guide. The class is steadily fixed, though not precisely limited; it is given, though not circumscribed; it is determined, not by a boundary line without, but by a central point within; not by what it strictly excludes, but by what it eminently includes; by an example, not by a precept; in short, instead of a Definition we have a Type for our director. (vol. 2, pp. 120-122)

Mill (1865) dropped the assumption of necessary and sufficient conditions, but he still assumed that types are defined by a set of features or characters stated in words. He weakened the requirements to a preponderance of defining characters:

> Whatever resembles the genus Rose more than it resembles any other genus, does so because it possesses a greater number of the characters of that genus, than of the characters of any other genus. Nor can there be the smallest difficulty in representing, by an enumeration of characters, the nature and degree of the resemblance which is strictly sufficient to include any object in the class. There are always some properties common to all things which are included. Others there often are, to which some things, which are nevertheless included, are exceptions. But the objects which are exceptions to one character are not exceptions to another: the resemblance which fails in some particulars must be made up for in others. The class, therefore, is constituted by the possession of all the characters which are universal, and most of those which admit of exceptions. (p. 277)

Both Whewell and Mill assume a range of variability in nature, but they propose different ways of measuring it. Instead of "a boundary line without," Whewell suggested "a central point within." But that criterion would require some measure of the distance between any instance and the type specimen. Instead of using a specimen, Mill defined his measure of similarity by enumerating the "characters" of a definition. In theory, Whewell's method is closer to nature, since it is based on a specimen taken from nature. In practice, both methods are based on words. Whewell uses descriptions of specimens, and Mill uses definitions abstracted from the descriptions. Whewell's method is one step closer to nature, but it depends on the words that biologists choose to describe nature.

The psychologist Eleanor Rosch wrote her bachelor's thesis on Wittgenstein's classification by family resemblance and her PhD dissertation on its psychological basis. Rosch and Mervis (1975) concluded that family resemblances characterize "prototype formation as part of the general process by which categories themselves are formed." They cited Zadeh (1965), but their analysis is closer to Whewell and Mill. They agree with Whewell that prototypes are the basis for classification. But they also give some support to Mill because the prototypes that people naturally choose are the ones that have the largest number of attributes or resemblances that characterize the category. These observations suggest that the cognitive basis for classification is a fuzzy kind of similarity, not rigid definitions or identity conditions. But if human thought is ultimately fuzzy, how is precise reasoning possible in science and mathematics?

Unlike Rosch and Mervis, who searched for a cognitive source of fuzziness, Immanuel Kant (1800) maintained that the open-ended variability of nature is the cause of fuzziness:

> Since the synthesis of empirical concepts is not arbitrary but based on experience, and as such can never be complete (for in experience ever new characteristics of the concept can be discovered), empirical concepts cannot be defined. Thus only arbitrarily made concepts can be defined synthetically. Such definitions... could also be called declarations, since in them one declares one's thoughts or renders account of what one understands by a word. This is the case with mathematicians.

In short, a precise definition is only possible when the author has complete control over the subject matter. But all authors control their subject to some extent. The critical questions are how, why, and to what extent.

2. Mathematical Language

Most mathematicians and logicians pay little attention to vagueness in ordinary language because their language is not vague. They are careful to use consistent definitions within a single document, but they often use different definitions in different documents. Therefore, mathematicians cite or restate the critical definitions and assumptions in every publication. Even the word *number*, the most fundamental in all of mathematics, has a long history of definitions that evolved over the centuries. Figure 2 shows an egg-yolk diagram for the many meanings of the word *number.*

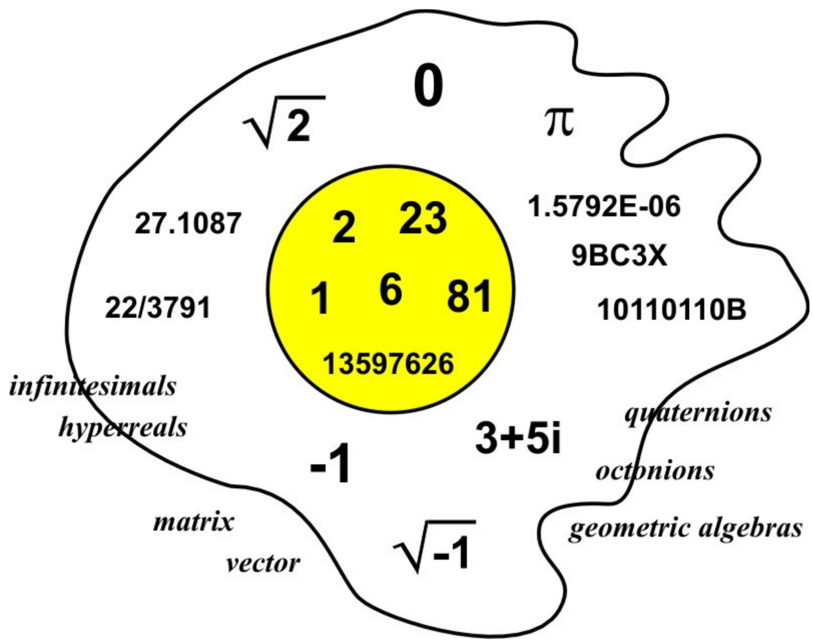

Figure 2. An egg-yolk diagram for the word *number*

The yolk of Figure 2 shows the positive integers, which were discovered or invented in prehistoric times. The egg white includes generalizations that can be mixed with the integers in the common arithmetic operations: rational numbers, irrational numbers, zero, negative numbers, and various encodings designed for computers. On the border or outside the egg are mathematical systems that share some of the mathematical operators, but with more variations. All these things called numbers have a kind of family resemblance, as Wittgenstein (1953 §67) said:

> Why do we call something a "number"? Well, perhaps because it has a – direct – relationship with several things that have hitherto been called number; and this can be said to give it an indirect relationship to other things we call the same name. And we extend our concept of number as in spinning a thread we twist fibre on fibre. And the strength of the thread does not reside in the fact that some one fibre runs through its whole length, but in the overlapping of many fibres.

Wittgenstein used the word *Sprachspiel* (language game) for various ways of using language. He compared the words of language to the pieces in a game of chess. The rules of chess are as precise as any version of mathematics, but some people define new rules that use the same pieces in a different way. In mathematics, the oldest games with numbers are counting, simple arithmetic, bookkeeping, and banking. But mathematicians have used the same symbols for different games, as Figure 2 illustrates. In each game, precision is possible because all the players agree to a fixed set of rules for using a fixed set of symbols or pieces. The word *number* has a precise meaning in each game, but when taken out of context, the word is ambiguous.

3. Relating Patterns to Patterns

When words are used to express novel experiences, they acquire new meanings or senses. But words seldom occur in isolation. They normally occur in larger patterns in which the senses of multiple words shift in a systematic way. Telephones, for example, led to new patterns for the words *talk, call,* and *conversation*. Cell phones enabled new patterns of activities, which led to further shifts in the senses of the words that express them. Smart phones combine those patterns with modified patterns of words for activities related to cameras, computers, GPS location, maps, games, television, and shopping. At each stage, old words are used in novel combinations, such as *cell phone* and *smart phone*. But even words that occur in the old lexical patterns acquire new senses from the novel activities they express.

In science, collections of patterns form theories. In other fields, they are called models, blueprints, project plans, or syndromes. Whatever they're called, collections of patterns are expressed in notations for which precision is important. Yet scientists are always aware of the experimental error, which they try to limit by carefully controlled experiments. Engineers express their frustration in a pithy slogan: All models are wrong, but some are useful. To bridge the gap between theories and the world, Figure 3 shows a model as a Janus-like structure, with an engineering side facing the world and an abstract side facing a theory.

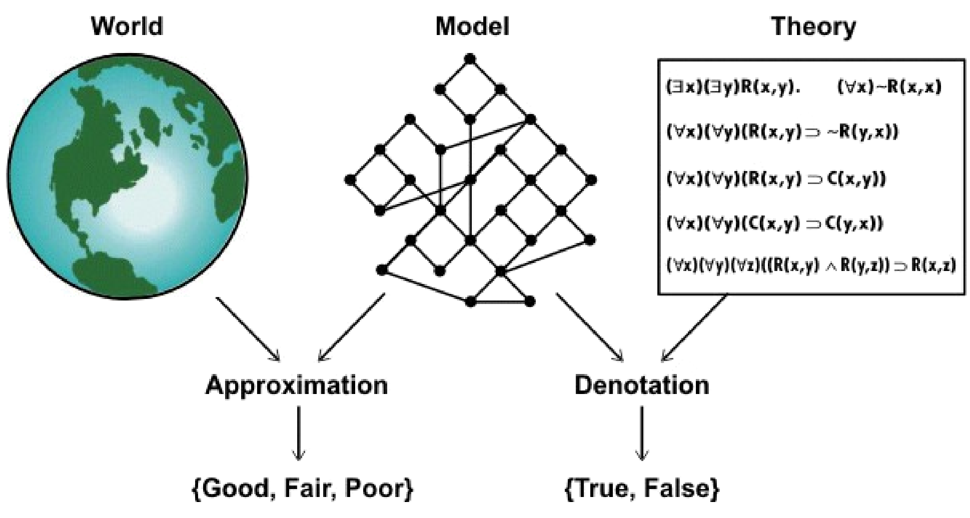

Figure 3: Relating a theory to the world

On the left is a picture of the physical world, which contains more detail and complexity than any humanly conceivable model or theory can represent. In the middle is a mathematical model that represents a domain of individuals **D** and a set of relations **R** over individuals in **D**. If the world had a unique decomposition into discrete objects and relations, the world itself would be a universal model, of which all accurate models would be subsets. But the selection of a domain and its decomposition into objects depend on the intentions of some agent and the limitations of the agent's measuring instruments. Even the best models are approximations to a limited aspect of the world for a specific purpose.

The two-stage mapping from theories to models to the world can reconcile a Tarski-style model theory with the fuzzy methods pioneered by Lotfi Zadeh. In Tarski's models, each sentence has only two possible truth values: {true, false}. In fuzzy logic, a sentence can have a continuous range of values from 0.0 for certainly false to 1.0 for certainly true. Hedging terms, such as likely, unlikely, very nearly true, or almost certainly false, represent intermediate values. The two-stage mapping of Figure 3 makes room for both kinds of reasoning: a rigorous two-valued logic for evaluating the truth of a mathematical theory in terms of a model; and a continuum of fuzzy values that measure the suitability of a particular model for a specific application. Such two-stage mappings have long been used in science and engineering: a strict two-valued logic for mathematical reasoning, and a continuum of values for quantifying experimental error and degree of approximation.

As Peirce said, "Logicians have too much neglected the study of vagueness, not suspecting the important part it plays in mathematical thought" (CP 5.505). In that same section, he said that the defining characteristic of a vague sentence is a violation of the law of contradiction: if the sentence *s* is vague, both *s* and *not s* can be true. Zadeh drew the following distinction (Mendel et al. 2010):

> Fuzzy relates to un-sharpness of class boundaries, while vagueness relates to insufficient specificity. As an illustration, "I'll be back in a few minutes" is fuzzy, but not vague. While "I'll be back sometime" is both fuzzy and vague... Usually, what is vague is fuzzy, but not vice-versa.

In practice, the word *sometime* often becomes *never.* With that qualification, Zadeh's examples are consistent with Peirce's criterion. But Peirce also distinguished vagueness from generality. For example, the general word *animal* is underspecified in comparison to *raccoon* or *beaver*, but it's not vague: the truth conditions for *animal* are just as precise as for *raccoon* or *beaver.*

In summary, Lotfi Zadeh should be congratulated for introducing a fruitful paradigm that has stimulated a large body of research with many valuable applications. The CWW methodology has introduced new ways of analyzing language and applying computable algorithms. But the discussions in the article by Mendel et al. (2010) show that CWW is unrelated to current linguistic research. More collaboration could help both fields clarify the sources of fuzziness.

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