Phaneroscopy and Hypoicons

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1. Phaneroscopy

For the field of phenomenology or *phaneroscopy*, Peirce coined the word *phaneron* as the equivalent of Aristotle's experiences in the psyche (pathêmata tês psychês, *On Interpretation*, 16a1-9). Peirce defined the phaneron as the "total of any one consciousness..., the sum of all we have in mind in any way whatever, regardless of its cognitive value. This is pretty vague: I intentionally leave it so... I do not limit the reference to an instantaneous state of consciousness; for the clause *in any way whatever* takes in memory and all habitual cognition" (EP 2:362, 1905). For any theory of cognition, reasoning, and communication, those vague ways require further analysis.

All empirical sciences depend on mathematics for their methods of reasoning and on phenomenology for their data. To relate mathematics and phenomenology to the empirical sciences, Figure 1 shows Peirce's classification of the sciences and the flow of information among them (EP 2.258-262, 1903). The dotted lines show two kinds of flow: theoretical dependencies flow from the upper left, but influences on the people who develop the theories may flow in either direction.

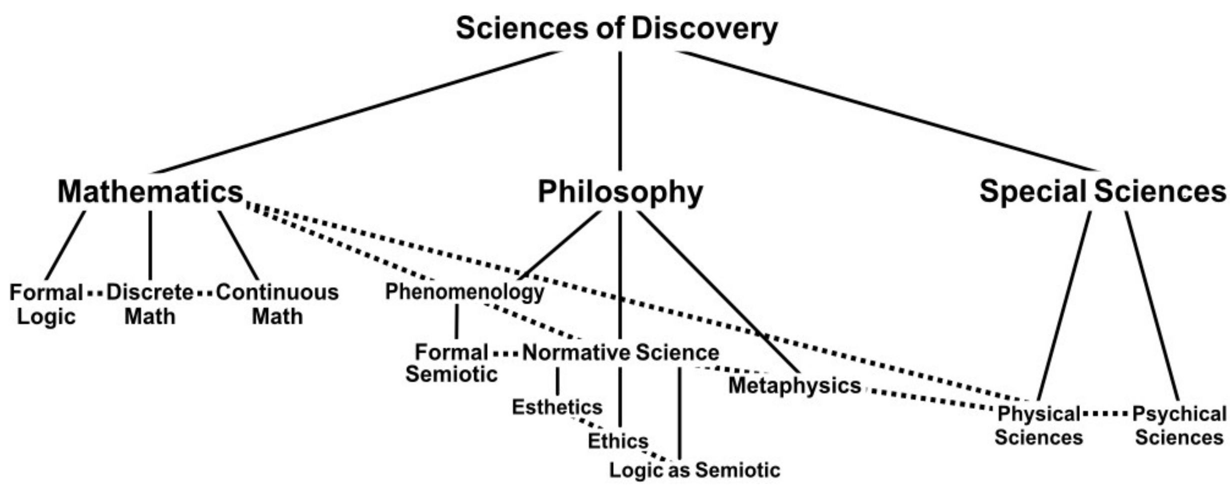

Figure 1: Flow of dependence and influence among the sciences

As Peirce wrote, "metaphysics ought to be founded on logic. To found logic on metaphysics is a crazy scheme" (CP 2.168, 1903). Instead, "the science of metaphysics must depend for its successful prosecution upon the recognition that a part if not all the conceptions which this science attaches to real objects are merely logical concepts put to different applications than that which logic makes of them; in consequence of which metaphysics depends to a very great degree upon the science of logic" (R602).

In Figure 1, formal logic is a branch of mathematics, and logic as semiotic is a branch of normative science. In modern contexts, the word logic without a qualifier usually means formal logic. But the 19th-century textbooks on logic included semiotic based on Aristotle and the Scholastics. In Peirce's writings, the meaning of logic without a qualifier depends on context. As the dotted lines in Figure 1 show, formal logic may be used in phaneroscopy, but the broader logic as semiotic may be used in metaphysics and the special sciences.

Phenomenology derives its data from perception, action, and feelings: "The elements of every concept enter into logical thought at the gate of perception and make their exit at the gate of purposive action; and whatever cannot show its passports at both those two gates is to be arrested as unauthorized by reason" (EP 2.241, 1903).

Phaneroscopy "is the science of the different elementary constituents of all ideas. Its material is, of course, universal experience, — experience I mean of the fanciful and the abstract, as well as of the concrete and real. Yet to suppose that in such experience the elements were to be found already separate would be to suppose the unimaginable and self-contradictory. They must be separated by a process of thought that cannot be summoned up Hegel-wise on demand. They must be picked out of the fragments that necessary reasonings scatter; and therefore it is that phaneroscopic research requires a previous study of mathematics." (R602, after 1903 but before 1908)

Formal semiotic is the result of using mathematics (arithmetic, geometry, graphs, and formal logic) to analyze the phaneron. That analysis produces the *phenomenological categories* of *Firstness*, *Secondness*, and *Thirdness*, which lead to the *trichotomy* of *icon*, *index*, and *symbol*. Further analysis of the "fragments" of "universal experience" led Peirce to subdivide icons in three kinds of *hypoicons*: *images, diagrams,* and *metaphors*:

> Any material image, as a painting, is largely conventional in its mode of representation; but in itself, without legend or label, it may be called a *hypoicon*. Hypoicons may roughly [be] divided according to the mode of Firstness which they partake. Those which partake the simple qualities, or First Firstnesses, are *images*; those which represent the relations, mainly dyadic, or so regarded, of the parts of one thing by analogous relations in their own parts, are *diagrams*; those which represent the representative character of a representamen by representing a parallelism in something else, are metaphors. (R478, EP 2:273-274, 1903)

Peirce also described diagrams as analogues "in respect to the dyadic relations of their parts to one another," and he called the third kind of hypoicons *examples* "in respect to their intellectual characters" (R478). He later wrote that diagrams "partake of a symbolic flavor" (R284, 1905).

In his writings on semiosis, Peirce used the word *interpretant* to emphasize the role of signs in communication. But Jappy (2011, 2019, 2020) showed that the two methods of analysis are complementary. The hypoicons of 1903 enable a "finer-grained structural analysis" of icons, while the trichotomies of 1908 address "the intentionality of semiosis." Structural analysis determines the options; intentionality determines the choice and sequence of options.

For reasoning, existential graphs (EGs) are refined hypoicons (Pietarinen 2011a, 2012). As a formal logic, EGs are a kind of mathematics. Therefore, EGs are available for reasoning about the phaneron. Logic as semiotic is the result of using formal logic, especially EGs, to analyze all aspects of perception, reasoning, and action in terms of value judgments of beauty, goodness, and truth.

2. Logic, Semiotic, and Truth

Aristotle defined the correspondence theory of truth: "To say of what is that it is not, or of what is not that it is, is false; but to say of what is that it is, or of what is not that it is not, is true" (1011b27). Ockham (1323) extended the theory to specify truth conditions for a subset of Latin. Both Peirce and Tarski adopted the Aristotle-Ockham theory, but they placed an abstract model between logic and the world. For Peirce, the model in the center of Figure 2 is an EG with no negations. Every such model is isomorphic to a Tarski model. But no model assembled from discrete components can be an exact representation of a continuous world.

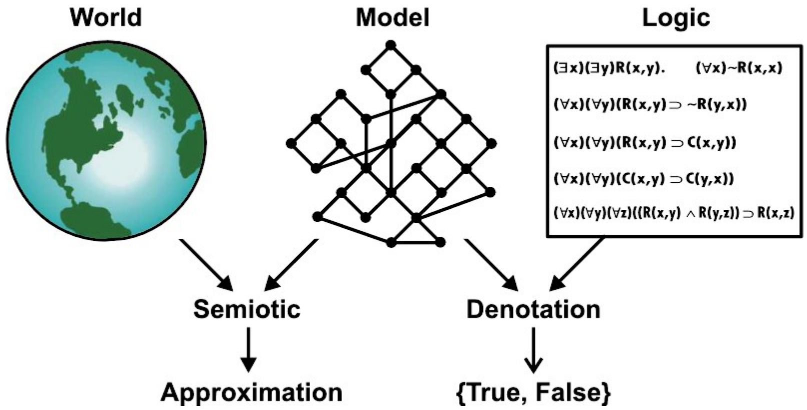

Figure 2. Models relate the world to language and logic

For the right side of Figure 2, Tarski's model theory or Peirce's endoporeutic would determine truth values for statements in a formal logic in terms of a formal model. Both systems would generate the same truth values for logically equivalent statements. For the left side of Figure 2, Peirce's logic as semiotic includes *methodeutic* for relating hypoicons that represent models to hypoicons derived by phaneroscopy.

Both Peirce and Frege rejected psychologism as a basis for developing formal logic. But as a logic of semiotic, Peirce used existential graphs to facilitate "imaging the otherwise nebulous, ghostlike, dubious abstractions of metaphysics" and endow "them with something of [the] distinctness of geometrical diagrams and with much of the convincingness of *working models*" (R298,1906).

In neuroscience, brain scans can detect the neural activations during phaneroscopy. The neuroscientist Damasio (2010) explicitly said that brains create "images, the main currency of our minds. Ultimately consciousness allows us to experience maps as images, to manipulate those images, and to apply reasoning to them." The psychologist Johnson-Laird (2002), who had written extensively about mental models, said that Peirce's existential graphs and rules of inference are a good candidate for a neural theory of reasoning:

> Peirce's existential graphs are remarkable. They establish the feasibility of a diagrammatic system of reasoning equivalent to the first-order predicate calculus. They anticipate the theory of mental models in many respects, including their iconic and symbolic components, their eschewal of variables, and their fundamental operations of insertion and deletion. Much is known about the psychology of reasoning... But we still lack a comprehensive account of how individuals represent multiply-quantified assertions, and so the graphs may provide a guide to the future development of psychological theories.

Diagrammatic reasoning is one of Peirce's most brilliant insights. His observations, quoted below, would be obvious to many mathematicians (Polya 1954). But they undermine the theories of Frege and the mainstream of 20th-century analytic philosophy:

> All necessary reasoning without exception is diagrammatic. That is, we construct an icon of our hypothetical state of things and proceed to observe it. This observation leads us to suspect that something is true, which we may or may not be able to formulate with precision, and we proceed to inquire whether it is true or not. For this purpose it is necessary to form a plan of investigation, and this is the most difficult part of the whole operation. We not only have to select the features of the diagram which it will be pertinent to pay attention to, but it is also of great importance to return again and again to certain features. (EP 2:212, 1903)

> The word *diagram* is here used in the peculiar sense of a concrete, but possibly changing, mental image of such a thing as it represents. A drawing or model may be employed to aid the imagination; but the essential thing to be performed is the act of imagining. Mathematical diagrams are of two kinds; 1st, the geometrical, which are composed of lines (for even the image of a body having a curved surface without edges, what is mainly seen by the mind's eye as it is turned about, is its generating lines, such as its varying outline); and 2nd, the algebraical, which are arrays of letters and other characters whose interrelations are represented partly by their arrangement and partly by repetitions. If these change, it is by instantaneous metamorphosis. (NEM 4:219)

> We form in the imagination some sort of diagrammatic, that is, iconic, representation of the facts, as skeletonized as possible. The impression of the present writer is that with ordinary persons this is always a visual image, or mixed visual and muscular... This diagram, which has been constructed to represent intuitively or semi-intuitively the same relations which are abstractly expressed in the premisses, is then observed, and a hypothesis suggests itself that there is a certain relation between some of its parts — or perhaps this hypothesis had already been suggested. In order to test this, various experiments are made upon the diagram, which is changed in various ways. (CP 2.778)

> Diagrammatic reasoning is the only really fertile reasoning. If logicians would only embrace this method, we should no longer see attempts to base their science on the fragile foundations of metaphysics or a psychology not based on logical theory. (CP 4.571)

For mathematical games like chess, diagrammatic reasoning is the essence of the game. Most chess experts can play a good blindfold game. For them, the board and pieces are the equivalent of Peirce's "drawing or model", which is a helpful, but optional aid to the imagination.

To study and compare the thought processes of mathematicians, Hadamard (1945) asked some of the most creative to answer a few questions. Their responses support the observations by Peirce and Polya. Einstein even used Peirce's words *visual* and *muscular*:

> The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. The psychical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be voluntarily reproduced and combined... The above-mentioned elements are, in my case, of visual and some of muscular type. Conventional words or other signs have to be sought for laboriously only in a secondary stage, when the mentioned associative play is sufficiently established and can be reproduced at will. (Einstein as quoted by Hadamard)

There is a continuum from counting sticks and drawings in the sand to the most elaborate mathematics and computer science. The methods of observation and imagination in mathematics apply to every branch of science, engineering, and common sense.

Peirce's semiotic is biomorphic, not anthropomorphic. From perception and action in the world, an agent, human or beast, would derive multiple hypoicons. A *mental model* of a *situation* may be defined as a systematic assembly of hypoicons derived by an agent at a particular time and place. An agent may be any living thing: "Mind has its universal mode of action, namely, by final causation. The microscopist looks to see whether the motions of a little creature show any purpose. If so, there is mind there" (CP 1.269, 1902). In R318 (1907), he wrote "The action of a sign generally takes place between two parties, the utterer and the interpreter. They need not be persons; ... many kinds of insect, and even plants make their livings by uttering signs."

For reasoning and communication, an agent may derive diagrams as abstractions from the hypoicons. For biosemiotic, Uexküll (1920) drew Figure 3 with German labels. The English labels relate it to the formal models by Peirce and Tarski and the mental models by psychologists.

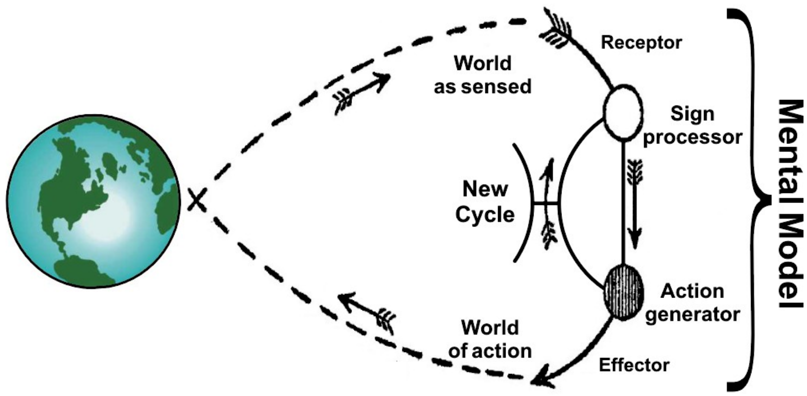

Figure 3: Uexküll's diagram for relating the world to a mental model

Peirce's foundation of logic and semiotic is sufficiently general to relate Tarski's formal models, Uexküll's biosemiotic, and modern cognitive science. In Figure 3, the cycle from the sign processor to the action generator and back again is the basis for diagrammatic reasoning in any living thing. The receptor receives images and feelings from any sensory modality and any organ in the body and brain. A quick stimulus-response would take milliseconds to relate a sensory hypoicon to a hypoicon that triggers a reaction. But repeated cycles could develop a sequence of hypoicons as *interpretants* for complex reasoning and communication in any branch of science.

For science and common sense, observation is the source of evidence about what exists. Imagination is the source of ideas about what can or should exist. As Peirce wrote, "When a man desires ardently to know the truth, his first effort will be to imagine what that truth can be. He cannot prosecute his pursuit long without finding that imagination unbridled is sure to carry him off the track. Yet nevertheless, it remains true that there is, after all, nothing but imagination that can ever supply him an inkling of the truth" (CP 1.46).

3. Phaneroscopy, Semiosis, and Action

The most fundamental semiotic problem is to characterize the kinds of transformations of signs between the receptor and the effector in Figure 3. Peirce's definition of hypoicon is sufficiently broad to include diagrams in any number of dimensions. That includes the linear notations for algebra and spoken languages and the multidimensional notations for signed languages, Euclid's diagrams, and various diagrams in science, engineering, and business. The twelve double arrows in Figure 4 illustrate the variety of phaneroscopic or semiotic transformations that involve hypoicons.

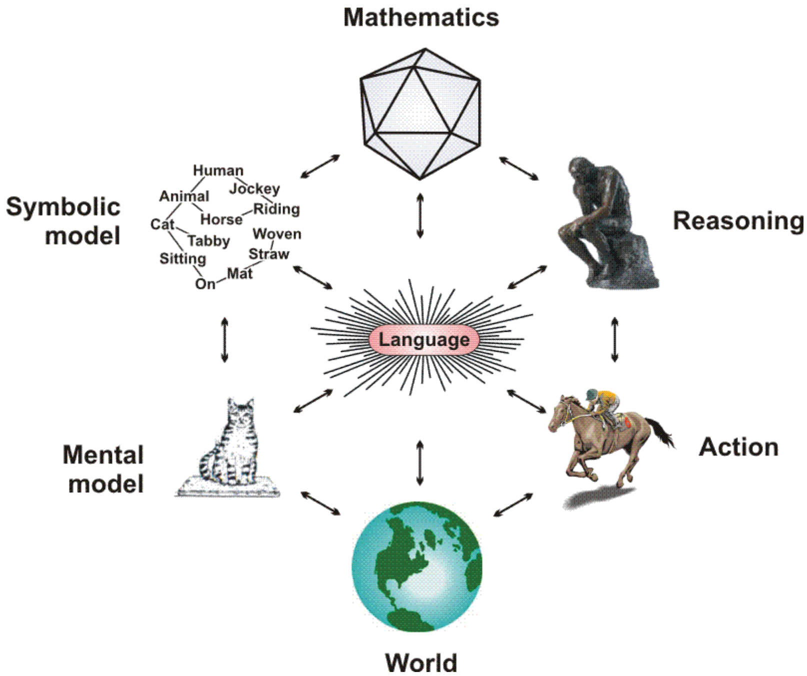

Figure 4: Semiotic transformations that involve hypoicons

The crystal at the top of Figure 4 represents the universe of all mathematical possibilities. It contains an uncountable infinity of patterns and all possible theories about them. The three arrows represent mappings to and from symbolic models (formal or informal), mappings to and from any kind of language (natural or artificial), and mappings to and from every possible method of reasoning (formal or informal).

The world at the bottom represents the universe of actualities, which includes the earth and its surroundings as they are perceived and acted upon by any living being. The three arrows represent the mappings of phaneroscopy, methodeutic, and action that relate the world to mental models, language, and life. The language node may represent any sign system that a species uses to communicate. Even bacteria form colonies with chemical signals for communication.

Peirce's universe of necessitants, which is not shown in Figure 4, may be represented in the same way as the other two universes of discourse. Various cultures give it different names: Laws of Nature, Logos, Dao, Dharma, Divine Mind, or Wisdom of the Ancients. Strict nominalists say that it doesn't exist. But whatever it's called, humans and beasts begin with an approximation based on instinct, early habits, and whatever they learn from their mothers.

[This section is incomplete. See Sowa (2013, 2015, 2018) for related issues.]

4. Observation and Imagination

Observation and imagination are the starting points for phaneroscopy. Observation is the source of evidence about what exists; imagination is the source of ideas about what may, should, or must exist. As Peirce wrote, "An imagination is an affection of consciousness which can be directly compared with a percept in some special feature and be pronounced to accord or disaccord with it" (CP 2.148).

Imagination is a kind of abduction, which must be tested by deduction, experiment, and analysis of the consequences. Peirce wrote "When a man desires ardently to know the truth, his first effort will be to imagine what that truth can be. He cannot prosecute his pursuit long without finding that imagination unbridled is sure to carry him off the track. Yet nevertheless, it remains true that there is, after all, nothing but imagination that can ever supply him an inkling of the truth" (CP 1.46).

For the three universes of discourse, observation provides evidence for actualities, imagination is the basis for discovering mathematical possibilities, and observation plus imagination determines the universe of necessities (laws and habits). Since an imagination "can be directly compared with a percept," it may be interpreted by the same hypoicons as perception. The hypoicons may also be combined, refined, and revised to generate new imaginations.

The dotted lines of Figure 1 show the influence of observation and imagination throughout the sciences, common sense, and everyday life. Since pure mathematics does not depend on observation, its theories can only be discovered by imagination. But the mathematicians who derive those theories are influenced by previous experiences derived from perception. A famous example is the eureka moment when Archimedes stepped in his bathtub. When he saw the water rise, he had a brilliant insight: the volume of water displaced by a body of any shape is equal to the volume of that body. That observation inspired him to imagine a new mathematical theory that solved an engineering problem: determine whether the king's crown is made of pure gold.

In mathematics, a theory is defined as a set of theorems derived or derivable as the deductive closure of a set of axioms. Two mathematical theories are identical (isomorphic) if and only if their sets of theorems are identical (have an exact one-to-one mapping). The iconic nature of the notation or proof procedures, which may be important for discovering or applying a theory, is irrelevant to the truth of any statement in a theory. In fact, the choice of axioms and primitive elements is also irrelevant. The set of theorems in classical first-order logic is identical for Frege's *Begriffsschrift* of 1879, Peirce's algebra of 1885, Peirce's existential graphs from 1897 to 1914, Whitehead and Russell's *Principia Mathematica* of 1910, and every notation since then.

Although pure mathematics does not depend on observation, the eureka example shows how the dotted lines in Figure 1 enable discoveries in empirical sciences to influence the mathematicians who develop new theories. Paul Halmos (1968), who wrote books that taught an entire generation of mathematicians, wrote:

> Mathematics — this may surprise or shock some — is never deductive in its creation. The mathematician at work makes vague guesses, visualizes broad generalizations, and jumps to unwarranted conclusions. He arranges and rearranges his ideas, and becomes convinced of their truth long before he can write down a logical proof... the deductive stage, writing the results down, and writing its rigorous proof are relatively trivial once the real insight arrives; it is more the draftsman's work not the architect's.

This comment by Halmos is consistent with the quotations by Peirce and Einstein in Section 2. The following remark by Peirce is also typical of many mathematicians:

> I do not think I ever *reflect* in words: I employ visual diagrams, firstly because this way of thinking is my natural language of self-communion, and secondly, because I am convinced that it is the best system for the purpose. (MS 619, 1909)

Note Peirce's emphasis on the word *reflect*. Like most people, he would think in words in an ordinary conversation. For deeper reflection, however, he used diagrams. Although words are necessary for language, the icons and hypoicons of phaneroscopy are the only basis for thinking by a pre-linquistic infant. Thinking in words channels thought in a fixed set of linguistic conventions. That is important for communication, but it can also restrict creativity. Children in an early stage of language learning are the most creative people in the world. "Out of the mouths of babes" comes baby talk – and sometimes pearls of wisdom.

Poets use language to express their art. But the best poets break through the old conventions to create new language patterns. As Robert Frost (1963) said, "I've often said that every poem solves something for me in life. I go so far as to say that every poem is a momentary stay against the confusion of the world.... We rise out of disorder into order. And the poems I make are little bits of order." With a change of the word *poem* to *theory*, Peirce and Einstein would agree.

In poetry, the iconic features of language, such as rhythm, rhyme, and patterns of sound, can stimulate the imagination of authors and their readers. In mathematics, the choice of notation is irrelevant to the truth of any statement, but the power of a good notation to stimulate the imagination, simplify the computation, and facilitate teaching and learning can be as important as the choice of words for a poet.

These observations are critical for the distinction between formal logic and logic as semiotic.

Although pure mathematics does not depend on observation, creative mathematicians rely on the same kinds of hypoicons as a creative child. But the iconic structure of a notation, which may be essential for stimulating the imagination, is irrelevant for its truth. As the deductive closure of a set of axioms, a formal theory does not depend on notation or iconicity. When multiple options for axioms or primitives produce the same deductive closure, the choice among them is irrelevant for pure mathematics, but it may be crucial for phaneroscopy and the sciences that depend on it.

5. Technology for Aiding Phaneroscopy

A quarter century before he coined the word *phaneroscopy*, Peirce (1878) published his most detailed report on phaneroscopy in scientific research: his book *Photometric Researches*. In the first paragraph of Chapter 1, he discussed the nature of light and the sensations it produces on the retina. On page 2. he distinguished *noumenal light* as "something in the external world" from *phenomenal light*, which is "considered as an appearance and as a function of the sensation." In relating phenomena to noumena, Peirce analyzed physical details about light and physiological details about human perception: "The different sensibility of the eye for the three primary colors also causes discrepancies in the observations of the relative brightness of differently colored stars, made by different observers or under different atmospheric circumstances or with telescopes of different power" (p. 6).

Other phaneroscopic issues involve instruments that aid perception, such as telescopes and "the colorimeter of Zöllner's astrophotometer." After comparing the results of the photometer to observations by 23 astronomers from Ptolemy to the 1870s, Peirce noted that his results "make stars more blue and less red than other observers" (p. 6). These quotations from the first two chapters of the book are a small sample of the issues Peirce considered. The remaining 180 pages show that phaneroscopy requires far more attention to detail and mathematical computation than the armchair philosophizing of the metaphysicians Peirce derided.

[This section is incomplete.]

For an overview of the ways Peirce anticipated and often surpassed innovations by his successors, see "Peirce's contributions to the 21st century" (Sowa 2006a). For relating metalanguage and diagrammatic reasoning to modal logic, see "Worlds, models, and descriptions" (Sowa 2006b). The icons in two-dimensional diagrams may be generalized to three dimensions and even 3+1 dimensions for motion and change. The technology of *virtual reality* can implement them. With VR goggles, people could wander through 3D EGs and manipulate the lines and shapes according to Peirce's rules. Reasoning in any notation requires analogies: the formal method of *unification* is a special case of analogy that is widely used in automated theorem provers.

To support analogies that map VR imagery to diagrams, the system of Cognitive Memory can find exact and approximate graph matches in logarithmic time (Majumdar & Sowa 2009, 2018). These methods of artificial intelligence, combined with Peirce's logic and semiotic, can serve as a foundation for a cognitive architecture that is compatible with the theories he developed in his late manuscripts. For robots, it can support a kind of *artificial phenomenology*: diagrammatic reasoning with Peirce's rules can map virtual reality to and from language and logic.

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